South East Asian J. of Mathematics and Mathematical Sciences
Vol. 21, No. 2 (2025), pp. 145-160

DOI: 10.56827/SEAJMMS.2025.2102.11 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

A NEW TYPE OF LORENTZIAN TRANS-SASAKIAN MANIFOLDS ADMITTING SEMI-SYMMETRIC NON-METRIC CONNECTION

Shadab Ahmad Khan, Nikhat Zulekha*, Anis Ahmad and Toukeer Khan**

Department of Mathematics & Statistics,
Integral University,
Lucknow - 226026, Uttar Pradesh, INDIA

E-mail : sakhan.lko@gmail.com, ch.anisahmad.13@gmail.com

*Department of Mathematics,
Trinity Institute of Innovation in Professional Studies,
Greater Noida - 201308, Uttar Pradesh, INDIA

E-mail : nikhatzulekha@gmail.com

**Department of Liberal Education,
Era University, Lucknow - 226003, Uttar Pradesh, INDIA

E-mail : toukeerkhan@gmail.com

(**Received:** *Jan. 17, 2025* **Accepted:** *Aug. 20, 2025* **Published:** *Aug. 30, 2025)*

Abstract: This research paper delves into the investigation of trans-Sasakian structures that accommodate a semi-symmetric non-metric connection on a manifold with a Lorentzian metric. Many significant results have been derived on such manifolds. The paper also explores conformally flat Lorentzian trans-Sasakian manifolds that admit semi-symmetric non-metric connections. Furthermore, explicit formulas for the curvature tensor, Ricci tensor, and Ricci operator are derived for three-dimensional Lorentzian trans-Sasakian manifolds with semi-symmetric non-metric connections.

Keywords and Phrases: η-Einstein manifold, conformally flat manifold, Lorentzian trans-Sasakian manifold, semi-symmetric non-metric connection.

2020 Mathematics Subject Classification: 53C15, 53C25, 53C40.

1. Introduction

Let M be an odd-dimensional manifold with the Riemannian metric g. It is well known that an almost contact metric structure (ϕ, ξ, η) (with respect to g) can be defined on M by a tensor field ϕ of type(1,1), a vector field ξ and a 1-form η. If M has a Sasakian structure (Kenmotsu structure), then M is called a Sasakian manifold (Kenmotsu manifold). Sasakian manifolds and Kenmotsu manifolds have been studied by several authors [1, 2, 12, 14, 23, 24].
In the classification of almost Hermitian manifolds by Gray and Hervella [11], there appears a class W_4 of Hermitian manifolds which are closely related to locally conformal Kaehler manifolds. An almost contact metric structure (ϕ, ξ, η, g) on trans-Sasakian structure M [17] if (MxR, J, G) belongs to the class W_4, where J is the almost complex structure on (MxR) defined by

$$J(X, f\frac{d}{dx}) = (\phi X - f\xi, \eta(X)\frac{d}{dx}),$$

for all vector fields $X \in \chi(M)$, where $\chi(M)$ is the Lie Algebra of smooth vector field on M, f is a smooth function on (MxR) and G is the product metric on (MxR). This may be expressed by condition [6]

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{1.1}$$

for smooth functions α and β in M. Hence we say that the trans-Sasakian structure is of type α and β. In particular, it is normal and it generalizes both α-Sasakian and β-Kenmotsu. From equation (1.1), we obtain

$$\nabla_X \xi = -\alpha(\phi X) - \beta(X - \eta(X)\xi) \tag{1.2}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y) \tag{1.3}$$

It is known that trans-Sasakian structures of type (0,0) , (0,β) and (α,0) are cosymplectic [5, 6]; β-Kenmotsu [13] and α-Sasakian [13] respectively. Several authors [2, 8, 14, 18, 20] have studied properties of Sasakian manifolds. [3, 7, 8, 15] have studied the structure of trans-Sasakian manifolds.

Let M be a differentiable manifold. When M has a Lorentzian metric g, that is, a symmetric non degenerate (0,2) tensor field of index 1, then M is called a Lorentzian manifold. Since the Lorentzian metric is of index 1, Lorentzian manifold M has not only space-like vector fields but also time-like and light-like vector fields. This difference with the Riemannian case gives interesting properties on the Lorentzian manifold. A differentiable manifold M has a Lorentzian metric if and only if M has a 1-dimensional distribution. Hence, odd dimensional manifold is able to have a Lorentzian metric. Therefore, it is a natural and interesting idea to define both a trans-Sasakian structure and a Lorentzian metric on an $(2n+1)$-dimensional manifold.

2. Lorentzian Trans-Sasakian Manifold

A $(2n+1)$-dimensional differential manifold M is called Lorentzian trans-Sasakian manifold if it has a (1,1) tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and the Lorentzian metric g which satisfy [7],

$$\phi^2 X = X + \eta(X)\xi, \tag{2.1}$$

$$\eta(\xi) = -1, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$g(X, \xi) = \eta(X), \phi\xi = 0, \eta(\phi X) = 0, \tag{2.4}$$

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(X)\phi X), \tag{2.5}$$

for all $X, Y \in T(M)$.
Also Lorentzian Trans-Sasakian manifold M satisfies

$$\nabla_X \xi = -\alpha(\phi X) - \beta(\phi^2 X), \tag{2.6}$$

$$(\nabla_X \eta)Y = \alpha g(\phi X, Y) + \beta g(\phi X, \phi Y), \tag{2.7}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

If $\alpha = 0$ and $\beta \in R$, the set of real numbers, then the manifold reduces to a Lorentzian β-Kenmotsu manifold studied by Yaliniz, Yildiz, and Turan [20]. If $\beta = 0$ and $\alpha \in R$, then the manifold reduces to a Lorentzian α-Sasakian manifold studied by Yildiz, Turan and Murathan [22]. If $\alpha = 0$ and $\beta = 1$, then the manifold reduces to Lorentzian Kenmotsu manifold. Furthermore, if $\alpha = 1$ and $\beta = 0$ then manifold reduces to Lorentzian Sasakian manifolds, this property was studied by Ikawa and Erdogan [12]. Also, Lorentzian para contact manifolds were introduced by [17] and further studied by [1]. Different structures on Lorentzian Sasakian manifolds are discussed by [4, 9, 10, 16, 19, 21]. Trans Lorentzian para-Sasakian manifolds have been used by Gill and Dube [10] and Lorentzian trans-Sasakian manifolds have been studied by De and De [7]. It is noticed that a $(2n+1)$-dimensional Lorentzian trans-Sasakian manifold M satisfies the following relations [7].

$$R(X,Y)\xi = (\alpha^2+\beta^2)(\eta(Y)X-\eta(X)Y) + 2\alpha\beta(\eta(Y)\phi X - \eta(X)\phi Y) \tag{2.8}$$

$$+(Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y,$$

$$\eta(R(X,Y)Z) = (\alpha^2+\beta^2)(g(X,Z)\eta(Y) - g(Y,Z)\eta(X)), \tag{2.9}$$

$$R(\xi,Y)\xi = (\alpha^2+\beta^2-\xi\beta)\phi^2 Y + (2\alpha\beta - \xi\alpha)\phi Y, \tag{2.10}$$

$$S(X,\xi) = (2n(\alpha^2+\beta^2)-\xi\beta)\eta(X) + (2n-1)(X\beta) - (\phi X)\alpha \tag{2.11}$$

$$+\psi(2\alpha\beta\eta(X) + X\alpha)$$

$$Q\xi = (2n(\alpha^2+\beta^2)-\xi\beta)\xi + (2n-1)grad\beta - \phi(grad\alpha) \tag{2.12}$$

$$+\psi(2\alpha\beta\xi + grad\alpha)$$

where R, S and Q are curvature tensor, Ricci curvature and Ricci operator given by

$$S(X,Y) = g(QX,Y)$$

and

$$\psi = \Sigma_{i=1}^{2n+1}\epsilon_i g(\phi e_i, e_i).$$

3. Existence of the Semi-Symmetric Non-Metric Connection on a Lorentzian Trans-Sasakian Manifold

Definition 3.1. *Let M be an (2n+1)-dimentional Lorentzian trans-Sasakian manifold. Then the linear connection $\bar{\nabla}$ defined on M as*

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X, \tag{3.1}$$

for all $X, Y \in \chi(M)$, is known as a semi-symmetric non-metric connection if it satisfies the equation (1.1) *and*

$$\bar{\nabla}_X g(Y,Z) = -\eta(Y)g(XZ) - \eta(Z)g(X,Y) \tag{3.2}$$

Next, we prove the existence of the semi- symmetric non -metric connection on an $(2n+1)$-dimensional Lorentzian trans-Sasakian manifold in the following theorem.

Theorem 3.2. *There exists a unique quarter-symmetric non-metric connection defined by* (3.1) *on an (2n+1)-dimentional Lorentzian trans-Sasakian manifold M.*

Proof. Suppose $\bar{\nabla}$ is the linear connection defined on an $(2n+1)-$ dimensional Lorentzian trans-Sasakian manifold M and is connected with ∇ by the relations

$$\bar{\nabla}_X Y = \bar{\nabla}_X Y + H(X,Y) \tag{3.3}$$

for all $X, Y \in \chi(M)$, where H denotes a tensor field of type (1, 2); using torsion tensor, equation (3.3) leads to

$$\bar{T}(X,Y) = H(X,Y) - H(Y,X) \tag{3.4}$$

for all $X, Y \in \chi(M)$.
Using equation (1.1), we have

$$g(H(X,Y),Z) - g(H(Y,X),Z) = \eta(Y)g(X,Z) - \eta(X)g(Y,Z) \tag{3.5}$$

From equation (3.3), we have

$$H(X,Y) = \bar{\nabla}_X Y - \nabla_X Y$$

$$g(H(X,Y),Z) = g(\bar{\nabla}_X Y - \nabla_X Y, Z)$$

Similarly,

$$g(H(X,Z),Y) = g(\bar{\nabla}_X Z - \nabla_X Z, Y)$$

Using above two equation, we have

$$g(H(X,Y),Z) + g(H(X,Z),Y) = g(\bar{\nabla}_X Y - \nabla_X Y, Z) + g(\bar{\nabla}_X Z - \nabla_X Z, Y)$$

Using (3.1) in above, we have

$$H^{'}(X,Y,Z) = \eta(Y)g(X,Z) + \eta(Z)g(X,Y)$$

Using (3.2) in above equation, we have

$$\bar{\nabla}_X g(Y,Z) = -H^{'}(X,Y,Z) \tag{3.6}$$

From equation (3.4) and (3.6), we can get

$$2g(H(X,Y),Z) = g(\bar{T}(X,Y),Z) + g(\bar{T}(Z,X,)Y) + g(\bar{T}(Z,Y),X) \tag{3.7}$$

$$-\bar{\nabla}_z g(X,Y) - \bar{\nabla}_Y g(X,Z)$$

Using (3.2) in (3.7), we have

$$2g(H(X,Y),Z) = g(\bar{T}(X,Y),Z) + g(\bar{T}(Z,X,)Y) + g(\bar{T}(Z,Y),X) \tag{3.8}$$

$$+2\eta(Z)g(X,Y)$$

As we have

$$g(\bar{T}(X,Y),Z) = \eta(Y)g(X,Z) - \eta(X)g(Y,Z) \tag{3.9}$$

Similarly, we can write

$$g(\bar{T}(Z,X),Y) = \eta(X)g(Z,Y) - \eta(Z)g(X,Y)$$

$$g(\bar{T}(Z,Y),X) = \eta(Y)g(Z,X) - \eta(Z)g(Y,X)$$

Using (3.8) and (3.9), we have

$$2g(H(X,Y),Z) = 2\eta(Y)g(X,Z)$$

Contracting both side, we have

$$H(X,Y) = \eta(Y)X \tag{3.10}$$

From (3.3) and (3.10), we have

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X,$$

for all $X, Y \in \chi(M)$.
Hence, the linear connection $\bar{\nabla}$ defined on an $(2n+1)$-dimensional Lorentzian trans-Sasakian manifold is a semi-symmetric non-metric connection. The converse part of the Theorem (3.2) is obvious. By covariant differentiation, we know that

$$(\bar{\nabla}_X \phi)Y = \bar{\nabla}_X \phi Y - \phi(\bar{\nabla}_X Y)$$

Using equation (3.1) in above, we get

$$(\bar{\nabla}_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X,Y)\xi - \eta(Y)\phi X) \tag{3.11}$$

From (3.1), putting $Y = \xi$ and using (2.6), we have

$$\bar{\nabla}_X \xi = -\alpha(\phi X) - (\beta+1)X - \beta\eta(X)\xi \tag{3.12}$$

Taking the covariant derivative of $\eta(X) = g(X,\xi)$ with respect to $\bar{\nabla}$ along the vector field X and Y, we get

$$(\bar{\nabla}_X \eta)Y = (\bar{\nabla}_X g)(Y,\xi) + g(\bar{\nabla}_X Y,\xi) + g(Y,\bar{\nabla}_X \xi) - \eta(\bar{\nabla}_X Y)$$

Using equations (3.1) and (3.2) in above, we have

$$(\bar{\nabla}_X\eta)Y = g(\nabla_X\xi, Y) + \eta(X)\eta(Y)$$

Using equation (2.6) in above, we have

$$(\bar{\nabla}_X\eta)Y = -\alpha g(\phi X, Y) - \beta(g(X, Y) + \eta(X)\eta(Y)) + \eta(X)\eta(Y)$$

Using equation (2.3) in above, we have

$$(\bar{\nabla}_X\eta)Y = -\alpha g(\phi X, Y) - \beta g(\phi X, \phi Y) + \eta(X)\eta(Y) \tag{3.13}$$

$$(\bar{\nabla}_X\eta)Y = -\alpha g(\phi X, Y) - \beta g(X, Y) - (\beta - 1)\eta(X)\eta(Y) \tag{3.14}$$

4. Some Important Results

Theorem 4.1. *In a Lorentzian trans-Sasakian manifold admitting semi-symmetric non-metric connection, we have*

$$\bar{R}(X, Y)\xi = (\alpha^2 + \beta^2 + 2)(\eta(Y)X - \eta(X)Y) + 2\alpha\beta(\eta(Y)\phi X - \eta(X)\phi Y) \tag{4.1}$$

$$+(Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y$$

Proof. As we know that

$$\bar{\nabla}_X\bar{\nabla}_Y Z = \nabla_X(\bar{\nabla}_Y Z) + \eta(\bar{\nabla}_Y Z)$$

$$\bar{\nabla}_X\bar{\nabla}_Y Z = \nabla_X\nabla_Y Z + \alpha g(\phi X, Z)Y + \beta g(X, Z)Y + (\beta - 1)\eta(X)\eta(Z)Y + \eta(\nabla_X Z)Y + \eta(Z)(\nabla_X Y) + \eta(\nabla_Y Z)X + \eta(Y)\eta(Z)X$$

Interchanging X and Y in above equation, we have

$$\bar{\nabla}_Y\bar{\nabla}_X Z = \nabla_Y\nabla_X Z + \alpha g(\phi Y, Z)X + \beta g(Y, Z)X + (\beta - 1)\eta(Y)\eta(Z)X + \eta(\nabla_Y Z)X + \eta(Z)(\nabla_Y X) + \eta(\nabla_X Z)Y + \eta(X)\eta(Z)Y$$

Now from equation (3.1), we have

$$\bar{\nabla}_{[X,Y]}Z = \nabla_{[X,Y]}Z + \eta(Z)[X, Y]$$

$$\bar{\nabla}_{[X,Y]}Z = \nabla_{[X,Y]}Z + \eta(Z)\nabla_X Y - \eta(Z)\nabla_Y X$$

Let $\bar{R}$ denote the curvature tensor with respect to the semi-symmetric non-metric connection, defined as

$$\bar{R}(X, Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z \tag{4.2}$$

$$\bar{R}(X,Y)Z = R(X,Y)Z + \alpha(g(\phi X,Z)Y - g(\phi Y,Z)X) + \beta(g(X,Z)Y - g(Y,Z)X) + (\beta-2)\eta(Z)(\eta(X)Y - \eta(Y)X)$$

Taking $Z=\xi$ in above equation, we have

$$\bar{R}(X,Y)\xi = R(X,Y)\xi + 2(\eta(X)Y - \eta(Y)X \tag{4.3}$$

Using (2.8) in (4.3), we have

$$\bar{R}(X,Y)\xi = (\alpha^2+\beta^2+2)(\eta(Y)X - \eta(X)Y) + 2\alpha\beta(\eta(Y)\phi X - \eta(X)\phi Y)$$

$$+(Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y$$

Lemma 4.2. *In a Lorentzian trans-Sasakian manifold admitting semi-symmetric non-metric connection, we have*

$$\eta(\bar{R}(X,Y)Z) = (\alpha^2+\beta^2 2)(\eta(X)g(Y,Z) - \eta(Y)g(X,Z)) \tag{4.4}$$

Proof. From (4.1), we have

$$g(\bar{R}(X,Y)\xi,Z) = (\alpha^2+\beta^2+2)(\eta(Y)g(X,Z) - \eta(X)g(Y,Z)) + 2\alpha\beta(\eta(Y)g(\phi X,Z) - \eta(X)g(\phi Y,Z)) + (Y\alpha)g(\phi X,Z) - (X\alpha)g(\phi Y,Z) + (Y\beta)g(\phi^2 X,Z) - (X\beta)g(\phi^2 Y,Z)$$

Interchanging ξ and Z in above, we have (4.4).

Lemma 4.3. *For a Lorentzian trans-Sasakian manifold admitting semi-symmetric non-metric connection, we have*

$$\bar{R}(\xi,Y)\xi = (\alpha^2+\beta^2+2-\xi\beta)\phi^2 Y + (2\alpha\beta+\beta-\xi\alpha)\phi Y \tag{4.5}$$

Proof. Replace X by ξ in (4.1), we get the result.

Corollary 4.4. *For a Lorentzian trans-Sasakian manifold admitting semi-symmetric non-metric connection, we have*

$$\bar{R}(\xi,\xi)\xi = 0.$$

Theorem 4.5. *In (2n+1) -dimentional Lorentzian trans-Sasakian manifolds admitting semi-symmetric non-metric connection, we have*

$$\bar{S}(X,\xi) = 2n(\alpha^2+\beta^2+2)\eta(X) + \psi[2\alpha\beta\eta(X) + X\alpha] \tag{4.6}$$

$$\bar{Q}\xi = [2n(\alpha^2+\beta^2+2) - \xi\beta]\xi + \psi[2\alpha\beta\xi + grad\alpha] \tag{4.7}$$

where $\bar{S}$ is the Ricci curvature and $\bar{Q}$ is the Ricci operator given by

$$\bar{S}(X,Y) = g(\bar{X},\bar{Y})$$

and

$$\psi = \Sigma_{i=1}^{2n+1}\epsilon_i g(\phi(e_i,e_i)$$

Proof. Let M be an $(2n+1)$-dimensional Lorentzian trans-Sasakian manifold admitting semi-symmetric non-metric connection. Then, the Ricci tensor $\bar{S}$ of the manifold M is defined by

$$\bar{S}(X,Y) = \Sigma_{i=1}^{2n+1}\epsilon_i(R(e_i,X)Y,e_i)$$

where $\epsilon_i = g(e_i,e_i), \epsilon_i = \pm 1$. From (4.1), we have

$$\bar{S}(X,\xi) = \Sigma_{i=1}\epsilon_i g[(\alpha^2+\beta^2+2)(\eta(X)e_i - \eta(e_i)X) + 2\alpha\beta(\eta(X)\phi e_i - \eta(e_i)\phi X) + (X\alpha)\phi e_i - (e_i\alpha)\phi X + (X\beta)\phi^2 e_i - (e_i\beta)\phi^2 X, e_i]$$

$$\bar{S}(X,\xi) = [2n(\alpha^2+\beta^2+2) - \xi\beta]\eta(X) + \psi[2\alpha\beta\eta(X) + X\alpha] - (\phi X)\alpha + (2n-1)(X\beta) \tag{4.8}$$

As, $\bar{S}(X,Y) = g(QX,Y)$,
This implies that $\bar{S}(X,\xi) = g(QX,\xi)$
Using the equation above in (4.8) and putting $X = \xi$, we have

$$\bar{Q}X = [2n(\alpha^2+\beta^2+2)-\xi\beta]\xi+(2n-1)(grad\beta)-\phi(grad\alpha)+\psi[(2\alpha\beta\xi+grad\alpha] \tag{4.9}$$

Using (4.8) and (4.9) we can obtain (4.6) and (4.7).

Remarks. *If in a (2n+1)-dimentional Lorentzian trans-Sasakian manifold of type (α,β) admitting semi-symmetric non-metric connection, we consider*

$$\phi(grad\alpha) = (2n-1)grad\beta,$$

then

$$\xi\beta = g(\xi, grad\beta) = \frac{1}{2n-1}g(\xi,\phi(grad\alpha)) = \frac{1}{2n-1}\eta(grad\alpha) = 0$$

and

$$X\beta = g(X, grad\beta) = \frac{1}{2n-1}g(X,\phi(grad\alpha)) = \frac{1}{2n-1}g(\phi X,(grad\alpha))$$

$$= \frac{1}{2n-1}(\phi X)\alpha$$

and hence using above condition in (4.8) *and* (4.9)*, we get* (4.6) *and* (4.7)*.*

Corollary 4.6. *For a Lorentzian trans-Sasakian manifold admitting semi- symmetric non-metric connection, we have*

$$\bar{S}(\xi,\xi) = \psi(\xi\alpha - 2\alpha\beta) - 2(\alpha^2 + \beta^2 + 2)$$

5. Conformally Flat Lorentzian Trans-Sasakian Manifolds admitting Semi-Symmetric Non-Metric Connection

In this section, we consider conformally flat Lorentzian trans-Sasakian manifold $M^{2n+1}(\phi,\xi,\eta,g),(n>1)$ admitting semi-symmetric non-metric connection $\bar{\nabla}$. The conformal curvature tensor $\bar{C}$ [7] is given by

$$\bar{C}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{(2n-1)}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y + g(Y,Z)\bar{Q}X \tag{5.1}$$

$$-g(X,Z)\bar{Q}Y] + \frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y].$$

where r is the scalar curvature of M. For a conformally flat manifold, we have $\bar{C}(X,Y)Z = 0$ for $n>1$ and hence from (5.1), we obtain

$$\bar{R}(X,Y)Z = \frac{1}{(2n-1)}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y + g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y] \tag{5.2}$$

$$-\frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y].$$

Replacing U by ξ in above equation, we get

$$\eta(\bar{R}(X,Y)Z) = \frac{1}{(2n-1)}[\bar{S}(Y,Z)\eta(X) - \bar{S}(X,Z)\eta(Y) + g(Y,Z)\bar{S}(X,\xi) \tag{5.3}$$

$$-g(X,Z)\bar{S}(Y,\xi)] - \frac{r}{2n(2n-1)}[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)].$$

Replacing Y by ξ in above, we have

$$\bar{S}(X,Z) = (2n-1)\eta(\bar{R}(X,\xi)Z) + g(X,Z)\bar{S}(\xi,\xi) - \eta(X)\bar{S}(\xi,Z) \tag{5.4}$$

$$-\eta(Z)\bar{S}(X,\xi) + \frac{r}{2n}[\eta(Z)\eta(X) + g(X,Z)].$$

Now, from equation (4.4), we have

$$\eta(\bar{R}(X,\xi)Z) = (\alpha^2+\beta^2+\alpha)[g(X,Z)+\eta(X)\eta(Z)] \tag{5.5}$$

Using equation (4.6), (5.4) and corollary 4.6 in equation (5.3) then we obtain

$$\bar{S}(X,Z) = [\frac{r}{2n} - (\alpha^2+\beta^2+2) + \psi(\eta\alpha - 2\alpha\beta)]g(X,Z) + [\frac{r}{2n} - (2n+1) \tag{5.6}$$

$$(\alpha^2+\beta^2+2) - 4\alpha\beta\psi]\eta(Z) - [\eta(X)(Z\alpha)+\eta(Z)(X\alpha)]\psi.$$

This provides the following results.

Theorem 5.1. *A conformally flat Lorentzian Trans-Sasakian manifold admitting semi-symmetric non-metric connection* $M^{2n+1}(\phi,\xi,\eta,g),(n>1)$ *is an* $\eta-$*Einstein manifold, provided* $\psi = trace\phi = 0$ *and* $\phi(grad\alpha) = (2n-1)grad\beta$.

6. Three - Dimentional Lorentzian Trans - Sasakian Manifolds admitting Semi -Symmetric Non-Metric Connection

Since the conformal curvature tensor vanishes in a three-dimensional Riemannian manifold, therefore from (5.2), we have

$$\bar{R}(X,Y,Z) = g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y + \bar{S}(Y,Z)X - \bar{S}(X,Z)Y \tag{6.1}$$

$$-\frac{r}{2}[g(Y,Z)X - g(X,Z)Y],$$

where $\bar{Q}$ is the Ricci operator, i.e., $g(\bar{Q}X,Y) = \bar{S}(X,Y)$ and r is the scalar curvature of the manifold.

Using the equation (4.8) and (4.9) in a three-dimentional Lorentzian trans - Sasakian manifold admitting semi-symmetric non-metric connection, we have

$$\bar{S}(X,\xi) = [2(\alpha^2+\beta^2+2) - \xi\beta]\eta(X) + \psi[2\alpha\beta\eta(X) + X\alpha] - (\phi X)\alpha + (X\beta) \tag{6.2}$$

$$\bar{Q}\xi = [2(\alpha^2+\beta^2+2) - \xi\beta]\xi + grad\beta - \phi(grad\alpha) + \psi[2\alpha\beta\xi + grad\alpha] \tag{6.3}$$

We deduce an expression for Ricci operator in a three-dimensional Lorentzian trans-Sasakian manifold admitting semi-symmetric non-metric connection in the following results.

Theorem 6.1. *In a 3-dimentional Lorentzian trans-Sasakian manifold admitting semi-symmetric non-metric connection Ricci operator is given by*

$$\bar{Q}X = [\frac{r}{2} + \xi\beta - (\alpha^2+\beta^2+2) + \psi(\xi\alpha - 2\alpha\beta)]X + [\frac{r}{2} + \xi\beta - 3(\alpha^2+\beta^2+2\alpha) \tag{6.4}$$

$$-4\alpha\beta\psi]\eta(X)\xi - [grad\beta - \phi(grad\alpha) + \psi(grad\alpha)]\eta(X) - [X\beta - (\phi X)\alpha + \psi(X\alpha)]\xi + [2\alpha\beta - \xi\alpha]\phi X$$

Proof. For a three-dimentional Lorentzian trans-Sasakian manifold admitting semi-symmetric non-metric connection, from (6.1), we have

$$\bar{R}(X,Y)\xi = \eta(Y)\bar{Q}X - \eta(X)\bar{Q}Y + \bar{S}(Y,\xi)X - \bar{S}(X,\xi)Y - \frac{r}{2}[\eta(Y)X - \eta(X)Y]$$

Using equation (6.2) in above equation, we obtain

$$\bar{R}(X,Y)\xi = \eta(Y)\bar{Q}X - \eta(X)\bar{Q}Y - [\frac{r}{2} + \xi\beta - 2(\alpha^2+\beta^2+2) - 2\alpha\beta\psi](\eta(Y)X - \eta(X)Y) + [(Y\alpha)\psi - (\phi Y)\alpha + (Y\beta)]X - [(X\alpha)\psi - (\phi X)\alpha + (X\beta)]Y.$$

In view of (4.1), substituting $Y = \xi$ and using (6.3), we have (6.4).

Corollary 6.2. *For a Lorentzian Sasakian manifold with a semi-symmetric non-metric connection, the Ricci operator is given by*

$$\bar{Q}X = [\frac{r}{2} - (\alpha^2+2) + \psi(\xi\alpha)]X + [\frac{r}{2} - 3(\alpha^2+2)]\eta(X)\xi + [\phi(grad\alpha) - \psi(grad\alpha)]\eta(X) + [(\phi X)\alpha - (X\alpha)\psi]\xi - (\xi\alpha)\phi X.$$

Corollary 6.3. *For a Lorentzian Kenmotsu manifold with a semi-symmetric non-metric connection, the Ricci operator is given by*

$$\bar{Q}X = [\frac{r}{2}\xi\beta - (\beta^2+2)]X + [\frac{r}{2} + \xi\beta - 3(\beta^2+2)]\eta(X)\xi - (grad\beta)\eta(X) - (X\beta)\xi.$$

Corollary 6.4. *For a Lorentzian cosymplectic manifold with a semi-symmetric non-metric connection, the Ricci operator is given by*

$$\bar{Q}X = (\frac{r}{2} - 2)X + (\frac{r}{2} - 6)\eta(X)\xi.$$

Corollary 6.5. *In a three-dimensional Lorentzian trans-Sasakian manifold admitting semi-symmetric non-metric connection, the curvature tensor R and Ricci tensor S are given by*

$$\bar{S}(X,Y) = [\frac{r}{2} + \xi\beta - (\alpha^2+\beta^2+2) + \psi(\xi\alpha - 2\alpha\beta)]g(X,Y) + [\frac{r}{2} + \xi\beta - 3(\alpha^2+\beta^2+2) - 4\alpha\beta\psi]\eta(X)\eta(Y) - \eta(X)[Y\beta - (\phi Y)\alpha + \psi(Y\alpha)] \tag{6.5}$$

$$-\eta(Y)[X\beta-(\phi X)\alpha+\psi(X\alpha)]+[(2\alpha+1)\beta-\xi\alpha]g(\phi X,Y)$$

$$\bar{R}(X,Y,Z)=[\frac{r}{2}+2\xi\beta-2(\alpha^2+\beta^2+2)+2\psi(\xi\alpha-2\alpha\beta)][g(Y,Z)X \tag{6.6}$$

$$-g(X,Z)Y]+g(Y,Z)[(\frac{r}{2}+\xi\beta-3(\alpha^2+\beta^2+2)-4\alpha\beta\psi)\eta(X)\xi+\eta(X)$$

$$(\phi(grad\alpha)-\psi(grad\alpha)-grad\beta)-(X\beta-(\phi X)\alpha+\psi(X\alpha))\xi]+g(X,Z)$$

$$[(\frac{r}{2}+\xi\beta-3(\alpha^2+\beta^2+2)-4\alpha\beta\psi)\eta(Y)\xi+\eta(Y)(\phi(grad\alpha)-\psi(grad\alpha)$$

$$-grad\beta)-(Y\beta-(\phi Y)\alpha+\psi(Y\alpha))\xi]+[(\frac{r}{2}+\xi\beta-3(\alpha^2+\beta^2+2)-4\alpha\beta\psi)$$

$$\eta(Y)\eta(Z)]+\eta(Y)((\phi Z)\alpha-\psi(Z\alpha)-Z\beta)-\eta(Z)(Y\beta-(\phi Y)\alpha+\psi(Y\alpha))]X$$

$$-[(\frac{r}{2}+\xi\beta-3(\alpha^2+\beta^2+2)-4\alpha\beta\psi)\eta(X)\eta(Z)+\eta(X)((\phi Z)\alpha-\psi(Z\alpha)-Z\beta)$$

$$-\eta(Z)(X\beta-(\phi X)\alpha+\psi(X\alpha))]Y+[2\alpha)\beta-\xi\alpha][g(\phi Y,Z)X-g(\phi X,Z)Y]$$

Proof. Taking the metric of equation (6.4) with respect to Y, we obtain equation (6.5). Then, again using equations (6.1), (6.4), and (6.5), we get (6.6).

Acknowledgement

The authors are thankful to Integral University, Lucknow, India, for providing manuscript communication number IU/R&D/2024-MCN0003292.

References

[1] Abood, H. M. and Mohammed, Y. A., A Study of New Class of Almost Contact Metric Manifolds of Kenmotsu Type, Tamkang Journal of Mathematics, 52(2) (2021), 253-266.

[2] Ahmad, M., Khan, S. A. and Khan, T., On Non-Invariant Hypersurfaces of a Nearly Hyperbolic Sasakian Manifold, International Journal of Mathematics, 28(8) (2017), 1750064-1-8.

[3] Bagewadi, C. S. and Venkatesha, Some Curvature Tensors on a trans-Sasakian Manifold, Turkish Journal of Mathematics, 31(2) (2007), 111-121.

[4] Barman, A., On Lorentzian α-Sasakian Manifolds Admitting a Type of Semi-Symmetric Non-Metric Connection, Palestine Journal of Mathematics, 9(2) (2020), 848-857.

[5] Blair, D. E., Contact manifolds in Riemannian geometry, Lecture Notes in Math, Springer Verlag, 1976.

[6] Blair, D. E. and Oubina, J. A., Conformal and related changes of metric on the product of two almost contact metric manifolds, Publicacions Matemátiques, 34(1) (990), 199-207.

[7] De, U. C. and De, K., On Lorentzian trans-Sasakian Manifolds, Commun. Fac. Sci. Univ. Ank Series A1, 62(2) (2013), 37-51.

[8] De, U. C. and Sarkar, A., On Three-dimensional trans Sasakian Manifolds, Extracta Mathematicae, 23(3) (2008), 265-277.

[9] Ghosh, G. and Dey, C., Generalised CR-submanifolds of a LP-Sasakian Manifolds, Filomat, 32(18) (2018), 6281-6287.

[10] Gill, H., and Dube, K. K., Generalized CR- Submanifolds of a trans Lorentzian para Sasakian manifold, Proc. Nat. Acad. Sci. India Sec. A Phys., 2 (2006), 119-124.

[11] Gray, A. and Harvella, L. M., The Sixteen Classes of almost Hermitian manifolds and their linear invariants, Ann. Math. Pura Appl., 123(4) (1980), 35-58.

[12] Ikawa, T., Erdogan, M., Sasakian manifolds with Lorentzian metric, Kyungpook Math. J., 35(3) (1996), 517-526.

[13] Janssen, D. and Vanhecke, L., Almost contact structures and curvature tensors, Kodai Math. J., 4(1) (1981), 1-27.

[14] Khan, S. A., Khan, T., Bilal, M. and Ahmad, A., A New Type of (ϵ)-Lorentzian Para-Sasakian Manifolds, Communications on Applied Nonlinear Analysis, 32(1s) (2024), 399-411.

[15] Marrero, J. C., The local structure of trans-Sasakian manifolds, Ann. Mat. Pura Appl., 4(162) (1992), 77-86.

[16] Matsumoto, K., Mihai, I. and Rosca, R., ξ-null geodesic gradient vector fields on a Lorentzian para-Sasakian manifold, J. Korean Math. Soc., 32 (1995), 17-31.

[17] Oubina, J. A., New classes of almost contact metric structures, Publicationes Mathematicae Debrecen, 32(4) (1985), 187-193.

[18] Prakasha, D. G., Prakash, A., Nagaraja, M. and Veersha, P., On (ϵ) - Lorentzian para-Sasakian Manifolds, Iranian Journal of Mathematical Sciences and Informatics, 17(2) (2022), 243-252

[19] Singh, G. and Sharma, V. K., Generalized CR-Submanifolds of Manifolds with a Lorentzian para Sasakian 3-Structure, International Journal of Mathematical Analysis, 16(2) (2022), 65-71.

[20] Yaliniz A, Yildiz A, Turan M., On three-dimensional Lorentzian $\beta-$ Kenmotsu manifolds, Kuwait Journal of Science and Engineering, 36(2A) (2009), 51-62.

[21] Yildiz, A., De, U. C. and Ata, E., On a type of lorentzian Para-Sasakian manifolds, Mathematical Reports, 16(1) (2014), 61-67.

[22] Yildiz, A., Turan, M. and Murathan, C., A class of Lorentzian α-Sasakian manifolds, Kyungpook Mathematical Journal, 49(4) (2009), 789-799.

[23] Siddiqi, M. D., On δ-Lorentzian trans-Sasakian manifold with semi-symmetric metric connection, Bol. Soc. Paran. Mat., 39 (2021), 113-135.

[24] Siddiqi, M. D., Chaubey, S. K. and Siddiqui, A. N., Clairaut anti-invarinat submersions from Lorentzian trans-Sasakian manifold, Arab Journal of Mathematical Sciences, 30(2) (2024), 134-149.

This page intentionally left blank.